STABLE BUNDLES ON HOPF MANIFOLDS

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Abstract. In this paper, we study holomorphic vector bundles on (diagonal) Hopf manifolds. In particular, we give a description of moduli spaces of stable bundles on generic (non-elliptic) Hopf surfaces. We also give a classification of stable rank-2 vector bundles on generic Hopf manifolds of complex dimension greater than two.

1. Introduction

In this paper, we study the stability properties of holomorphic vector bundles on diagonal Hopf manifolds. Recall that a Hopf manifold is defined as the quotient of the punctured n-space $\mathbb{C}^n \backslash \{0\}$ by an infinite cyclic group, generated by a contraction of $(\mathbb{C}^n, 0)$. If the contraction is multiplication by a diagonal matrix, then the Hopf manifold is called *diagonal*. All Hopf manifolds are non-algebraic. In particular, every diagonal Hopf manifold is diffeomorphic to $S^1 \times S^{2n+1}$, implying that it is non-Kählerian. A generic Hopf manifold possesses very few curves, the only ones being n elliptic curves corresponding to the coordinate axes in $\mathbb{C}^n$. But there exist, nevertheless, Hopf manifolds with infinite families of curves. For example, if the contraction defining the manifold is a multiple of the identity, then the manifold admits an elliptic fibration.

Holomorphic vector bundles on elliptically fibred Hopf manifolds are by now well-understood. In the case of surfaces, these bundles have been completely classified and a detailed analysis of their moduli spaces can be found in [Mo1, BrMo3]. Moreover, for Hopf manifolds of dimension greater than two, the question has been settled by Verbitsky in [V1], where he studies stable bundles on positive principal elliptic fibrations. More generally, he proves that on a (possibly non-elliptic) diagonal Hopf manifold of dimension greater than two, any coherent sheave $\mathcal{F}$ is *filtrable*, that is, admits a filtration by a sequence of coherent sheaves

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_r = \mathcal{F}$$

with $\operatorname{rk} \mathcal{F}_i/\mathcal{F}_{i-1} \leq 1$ [V1, V2].

Less is known about the classification and stability properties of bundles on generic (non-elliptic) Hopf manifolds. Although some partial results have been obtained in this direction by Mall [Ma2, Ma3], his study has focused only on vector bundles whose pullback to the universal cover $\mathbb{C}^n \backslash \{0\}$ is holomorphically trivial (such bundles are given by factors of automorphy). In particular, he proves that the pullback of a vector bundle on a Hopf manifold to $\mathbb{C}^n \backslash \{0\}$ is holomorphically trivial if and only if it possesses a filtration by vector bundles, giving in the process a partial classification such bundles on generic Hopf manifolds. However, as shown in

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this paper, most vector bundles on diagonal Hopf manifolds do not admit filtrations by vectors bundles; in fact, on Hopf surfaces, vector bundles are generically non-filtrable.

The paper is organised as follows. We begin by recalling some definitions and topological properties of Hopf manifolds; holomorphic vector bundles are described in sections three and four. For simplicity, we restrict our presentation on generic Hopf manifolds to rank-2 vector bundles; nevertheless, similar results holds for vector bundles of arbitrary rank. The third section of the paper is devoted to the study of bundles on surfaces. We begin by proving that, on a diagonal Hopf surface, holomorphic vector bundles possess a filtration by vector bundles if and only if they are topologically trivial; bundles with non-trivial second Chern classes are, however, generically non-filtrable and therefore stable. We then give a classification of stable filtrable bundles. Moduli spaces of stable bundles on diagonal Hopf surfaces admit natural Poisson structures; we describe their associated symplectic leaves for elliptic Hopf surfaces. Note that in the elliptic case, the moduli also admit the structure of algebraically completely integrable Hamiltonian systems [Mo1]. Finally, stability conditions for vector bundles on generic Hopf manifolds of dimension greater than two are given in the last section; in particular, we show that there exist stable bundles on them.

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2. Preliminaries

A diagonal Hopf manifold X is defined as the quotient of $\mathbb{C}^n_* := \mathbb{C}^n \backslash \{0\}$ by the cyclic group generated by a contraction of $(\mathbb{C}^n, 0)$ of the form

$$\begin{array}{rcl} \mu : \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ (z_1, \ldots, z_n) & \mapsto & (\mu_1 z_1, \ldots, \mu_n z_n), \end{array}$$

where $\mu_1, \ldots, \mu_n$ are complex numbers such that $0 < |\mu_1| \leq |\mu_2| \leq \cdots \leq |\mu_n| < 1$. Every diagonal Hopf manifold is diffeomorphic to $S^1 \times S^{2n+1}$. Moreover, its Hodge numbers are all zero except for $h^{0,0}_X = h^{0,1}_X = h^{n+1,n}_X = h^{n+1,n+1}_X = 1$ [Ma1].

2.1. **Notation.** We begin by fixing some notation.

- Denote by $p : \mathbb{C}^n_* \to X$ the canonical projection map.
- $U_i := \{z_i \neq 0\}$ is an open subset of $\mathbb{C}^n_*$ for all $i = 1, \ldots, n$.
- $X_i := p(U_i)$ is an open subset of X for all $i = 1, \ldots, n$.
- $H_i := p(\{z_i = 0\})$ is a hypersurface in X, for all $i = 1, \ldots, n$, that is isomorphic to the Hopf manifold of dimension $n-1$ corresponding to the diagonal matrix $(\mu_1, \ldots, \hat{\mu}_i, \ldots, \mu_n)$.
- $H_{k_i k_j} := p(\{z_i^{k_i} = z_j^{k_j} = 0\})$, for $k_i, k_j \geq 0$ and $0 \leq i, j \leq n$ such that $i \neq j$, is a codimension 2 subvariety of X.
- $T_i := p(\{(0, \ldots, z_i, \ldots, 0)\})$ is the elliptic curve $\mathbb{C}^*/\mu_i$ for all $i = 1, \ldots, n$. Note that $H_i \cap T_i = \emptyset$ for all $i = 1, \ldots, n$. Moreover, $T_i \cap T_j = \emptyset$, if $i \neq j$.

2.2. **Classical Hopf manifolds.** A diagonal Hopf manifold is called *classical* if $\mu_1 = \mu_2 = \cdots = \mu_n = \mu$. These manifolds admit a natural holomorphic elliptic fibration

$$\begin{array}{rcl} \pi : X & \to & \mathbb{P}^n \\ (z_1, \cdots, z_n) & \mapsto & [z_1 : \cdots : z_n], \end{array}$$

with fibre the elliptic curve $T = \mathbb{C}^*/\mu$. Moreover, the relative Jacobian of $X \xrightarrow{\pi} \mathbb{P}^n$ is isomorphic to

$$J(X) = \mathbb{P}^n \times T^* \xrightarrow{p_1} \mathbb{P}^n,$$

where T^* denotes the dual elliptic curve determined by a non-canonical identification $T^* := \mathrm{Pic}^0(T) \cong T$.

2.3. **Generic Hopf manifolds.** A diagonal Hopf manifold is called *generic* if there are no non-trivial relations between the μ_i's of the form

$$\prod_{i \in A} \mu_i^{r_i} = \prod_{j \in B} \mu_j^{r_j},$$

where $r_i, r_j \in \mathbb{N}$, $A \cap B = \emptyset$, and $A \cup B = \{1, \ldots, n\}$. It is important to note that a generic Hopf manifold only contains n irreducible curves, namely the images $T_1, \ldots, T_n$ of the punctured z_1-, …, z_n-axes (see [BPV] for the case $n = 2$). Although these curves are elliptic, the manifold itself does not admit an elliptic fibration. Moreover, given that there are no relations between the μ_i's, the H_i's are the only irreducible hypersurfaces in X.

2.4. **Hopf surfaces.** Diagonal Hopf surfaces can be divided into four categories: classical, generic, resonant, and hyperresonant. The last two are defined by diagonals (μ_1, μ_2) such that $\mu_1^p = \mu_2^q$ for some integers p and q (the resonant case corresponds to $p = 1$). Note that a (hyper)resonant surface X admits an elliptic fibration $\pi : X \to \mathbb{P}^1$ with singular fibres the curve(s) T_1 (and T_2). However, it can be covered by the classical Hopf surface given by diagonal $(\mu_2^{1/p}, \mu_2^{1/p})$, whose elliptic fibration does not have singular fibres. Vector bundles on the (hyper)resonant surface X can therefore be studied by analysing their pullback to this classical Hopf surface [BrMo1, BrMo2, BrMo3].

In general, Hopf surfaces are defined as compact complex surfaces that admit $\mathbb{C}^*$ as a universal covering. Although every diagonal Hopf surface is diffeomorphic to $S^1 \times S^3$, there exist many Hopf surfaces that are not. Examples of such Hopf surfaces can be constructed as follows. For any integer d, let Θ_d^* denote the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(d)$ minus the zero section; with respect to this notation, we have $\mathbb{C}^2_* \cong \Theta_{-1}^*$. We cover Θ_d^* by the open sets U_0 and U_∞ with coordinates (z, t) and $(\xi, s) = (z^{-1}, z^{-d}t)$, respectively. Given complex number μ_1 and μ_2, with $0 < |\mu_1| \le |\mu_2| < 1$, and β such that $\beta^d = \mu_1\mu_2^{-1}$, we define the following $\mathbb{Z}$-action on Θ_d^*: $(z, t) \mapsto (\beta z, \mu_1 t)$ on U_0, and $(\xi, s) \mapsto (\beta^{-1}\xi, \mu_2 s)$ on U_∞. The quotient

$$X := \Theta_d^*/(\beta, \mu_1)$$

is a Hopf surface, which is non-primary when $d \neq -1$. Note that if $d = -1$, then we simply have a diagonal Hopf surface:

$$\mathbb{C}^2_*/(\mu_1, \mu_2) \cong \Theta_{-1}^*/(\mu_1^{-1}\mu_2, \mu_1).$$

Moreover, if $\mu_1 = \mu_2$ and $\beta = 1$, then X is a principal elliptic fibration over $\mathbb{P}^1$ with fibre $\mathbb{C}^*/\mu_1$. These Hopf surfaces are diffeomorphic to $S^1 \times C_d$, where C_d is

the S^1-bundle over S^2 with Chern class d. Consequently, we have $H^i(X,\mathbb{Z}) = \mathbb{Z}$, for $i = 0, 1, 4$, $H^2(X,\mathbb{Z}) = \mathbb{Z}_{|d|}$, and $H^3(X,\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_{|d|}$.

2.5. **Line bundles.** The only divisors that exist on a generic Hopf manifold X are linear combinations of the hypersurfaces $H_1, \dots, H_n$:

$$\mathrm{Div}(X) = \{m_1 H_1 + \cdots + m_n H_n\} = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z},$$

with the relations $H_i \cdot H_j = 0$; the canonical divisor is $K_X = -H_1 - \cdots - H_n$. Whereas divisors on a classical Hopf manifold X are pullbacks of hypersurfaces on $\mathbb{P}^{n-1}$. In particular, the canonical divisor is $K_X = \pi^* K_{\mathbb{P}^{n-1}}$.

Although Hopf manifolds have few divisors, there are many line bundles on them. For example, on any diagonal Hopf manifold X, we have $\mathrm{Pic}(X) = \mathrm{Pic}^0(X) = \mathbb{C}^*$: line bundles correspond to constant factors of automorphy. The line bundle given by the factor $a \in \mathbb{C}^*$ is constructed by taking the quotient of the trivial line bundle $\bar{\mathbb{C}}$ on $\mathbb{C}^n_*$ by the following $\mathbb{Z}$-action:

$$\begin{array}{ccc} \mathbb{C}^n_* \times \mathbb{C} & \to & \mathbb{C}^n_* \times \mathbb{C} \\ (z,t) & \mapsto & (\mu z, at). \end{array}$$

From now on, we shall denote by L_a the line bundle corresponding to the factor a. Note that the restriction of $\mathcal{O}_X(H_i)$ to the elliptic curve $T_i = \mathbb{C}^*/\mu_i$ is trivial, so that $\langle \mathcal{O}_X(H_i)\rangle = \mathbb{Z}$ is in the kernel of the natural restriction map $\mathrm{Pic}(X) \xrightarrow{r} \mathrm{Pic}^0(T_i) \to 0$. Consequently, we have $\mathcal{O}_X(H_i) = L_{\mu_i}$ for all $i = 1, \dots, n$.

The cohomology of line bundles on classical and generic Hopf manifolds the following [Ma1]. Given a line bundle L_a on the Hopf manifold X, we denote

$$h^{p,0} := h^p(X, L_a).$$

For a classical Hopf manifold X, given by the diagonal $(\mu, \dots, \mu)$, we have:

$$h^{p,0} = \begin{cases} h^p(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) & \text{if } a = \mu^m \text{ for some integer } m, \\ 0 & \text{otherwise.} \end{cases}$$

We now consider a generic Hopf manifold X, given by the diagonal $(\mu_1, \dots, \mu_n)$. If $n = 2$, then given the line bundle L_a on the generic Hopf surface X, we have:

$$h^{0,0} = \begin{cases} 1 & \text{if } a = \mu_1^{m_1}\mu_2^{m_2}, \text{ with } m_1, m_2 \geq 0, \\ 0 & \text{otherwise;} \end{cases}$$

$$h^{1,0} = h^{0,0} + h^{2,0};$$

$$h^{2,0} = \begin{cases} 1 & \text{if } a = \mu_1^{m_1}\mu_2^{m_2}, \text{ with } m_1, m_2 < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, for $n \geq 3$, the cohomology groups of the line bundle L_a are the following:

$$h^{0,0} = h^{1,0} = \begin{cases} 1 & \text{if } a = \mu_1^{m_1} \dots \mu_n^{m_n}, \text{ with } m_1, \dots, m_n \geq 0, \\ 0 & \text{otherwise;} \end{cases}$$

$$h^{p,0} = 0 \text{ if } 2 \leq q \leq n-2;$$

$$h^{n-1,0} = h^{n,0} = \begin{cases} 1 & \text{if } a = \mu_1^{m_1} \dots \mu_n^{m_n}, \text{ with } m_1, \dots, m_n < 0, \\ 0 & \text{otherwise.} \end{cases}$$

2.6. **Finite coverings.** In this section, we describe (finite) cyclic coverings of Hopf surfaces. We begin by noting that elliptically fibred Hopf surfaces admit many cyclic coverings as they correspond to pullbacks of cyclic coverings on $\mathbb{P}^1$ via the projection π. This is certainly not the case for generic Hopf surfaces. Let us consider a generic Hopf surface X given by the diagonal (μ_1, μ_2). Let $\varphi : Y \to X$ be the r-cyclic covering of X branched along the smooth divisor B on X and determined by the line bundle $\mathcal{L} \to X$, where $\mathcal{O}_X(B) = \mathcal{L}^{\otimes r}$ (see [BPV, F]). The smooth variety Y is then defined as the subvariety of the total space of the line bundle $\mathcal{L}$ given by

$$Y := \{y \ : \ y^r = s\},$$

where s is a global section of $\mathcal{L}^{\otimes r}$ with effective divisor $(s) = B$. However, since X is a generic Hopf surface, we have $h^0(X, \mathcal{O}_X(B)) = 1$, implying that there is a unique cyclic covering corresponding to each $\mathcal{L}$.

Lemma 2.1. *Suppose that $X = \mathbb{C}^2_*/(\mu_1, \mu_2)$ is a generic Hopf surface. An r-cyclic covering of X is then either a generic Hopf surface or a non-elliptic non-primary Hopf surface of the form $\Theta^*_{-r}/(\beta, \mu_1)$, with $\beta^r = \mu_1$ (see section 2.4 for notation).*

Proof. On a generic Hopf surface, there are only four possibilities for the effective reduced divisor B, namely, $B = 0,\ T_1,\ T_2$, or $T_1 + T_2$, with associated line bundles $\mathcal{O}_X(B) = \mathcal{O}_X,\ L_{\mu_2},\ L_{\mu_1}$, and $L_{\mu_1\mu_2}$, respectively. We treat each case seperately.

If $B = 0$, then the line bundle $\mathcal{L}$ is an r-th root of unity of order k, where k is an integer that divides r. If $k = 0$, that is, $\mathcal{L} = \mathcal{O}_X$, then Y is a disconnected surface made up of r copies of X. If $\mathcal{L}$ has instead order r, then Y is a connected smooth surface that is an unramified r-to-one covering of X. In particular, there is a unique unramified double cover that is the Hopf surface $Y := \mathbb{C}^2_*/(\mu_1^2, \mu_2^2)$ with projection onto X given by $(z_1, z_2) \mapsto (z_1, z_2)$. Finally, if the order of $\mathcal{L}$ is $k \neq 0, m$, then $m = kl$ for some integer l and Y is disconnected surface made up of l copies of an unramified k-to-one cover Y' of X.

If $B = T_1$, then the line bundle $\mathcal{L}$ is given by a factor of automorphy $\alpha \in \mathbb{C}^*$ such that $\alpha^r = \mu_2$. The induced r-to-one cover of X is then the Hopf surface $Y := \mathbb{C}^2_*/(\mu_1, \alpha)$ with projection onto X given by $(z_1, z_2) \mapsto (z_1, z_2^r)$. Similarly, one sees that the r-to-one cover of X determined by $B = T_2$ are Hopf surfaces of the form $Y := \mathbb{C}^2_*/(\alpha, \mu_2)$ with $\alpha^r = \mu_1$.

Finally, consider $B = T_1 + T_2$; choose $\beta \in \mathbb{C}^*$ such that $\beta^r = \mu_1^{-1}\mu_2$. Using the notation of section 2.4, we have $X = \Theta^*_{-1}/(\mu_1^{-1}\mu_2, \mu_1)$ and its r-to-one cover is the Hopf surface $Y := \Theta^*_{-m}/(\beta, \mu_1)$ with projection onto X given by $(z, t) \mapsto (z^r, t)$. $\square$

2.7. **Degree and stability.** The degree of a vector bundle can be defined on any compact complex manifold M. Let $d = \dim_{\mathbb{C}} M$. A theorem of Gauduchon's [G] states that any hermitian metric on M is conformally equivalent to a metric, called a *Gauduchon metric*, whose associated (1,1) form ω satisfies $\partial\bar{\partial}\omega^{d-1} = 0$. Suppose that M is endowed with such a metric and let L be a holomorphic line bundle on M. The *degree of L with respect to ω* is defined [Bh], up to a constant factor, by

$$\deg L := \int_M F \wedge \omega^{d-1},$$

where F is the curvature of a hermitian connection on L, compatible with $\bar{\partial}_L$. Any two such forms F differ by a $\partial\bar{\partial}$-exact form. Since $\partial\bar{\partial}\omega^{d-1} = 0$, the degree is independent of the choice of connection and is therefore well defined. This notion of degree is an extension of the Kähler case. If M is Kähler, we get the usual

topological degree defined on Kähler manifolds; but in general, this degree is not a topological invariant, for it can take values in a continuum (see below).

Having defined the degree of holomorphic line bundles, we define the *degree* of a torsion-free coherent sheaf $\mathcal{E}$ on M by

$$\deg(\mathcal{E}) := \deg(\det \mathcal{E}),$$

where $\det \mathcal{E}$ is the determinant line bundle of $\mathcal{E}$, and the *slope of* $\mathcal{E}$ by

$$\mu(\mathcal{E}) := \deg(\mathcal{E})/\mathrm{rk}(\mathcal{E}).$$

The notion of stability then exists for any compact complex manifold:

A torsion-free coherent sheaf $\mathcal{E}$ on M is stable *if and only if for every coherent subsheaf $\mathcal{S} \subset \mathcal{E}$ with $0 < rk(\mathcal{S}) < rk(\mathcal{E})$, we have $\mu(\mathcal{S}) < \mu(\mathcal{E})$.*

Remark 2.2. With this definition of stability, many of the properties from the Kähler case hold. For example, all line bundles are stable whereas decomposable bundles are always unstable. In addition, for rank two vector bundles on a surface, it is sufficient to verify stability with respect to line bundles; in particular, if such a bundle is non-filtrable, then it is automatically stable. Finally, if a vector bundle E is stable, then it is simple, that is, $h^0(M, End(E)) = 1$.

Example 2.3. Let X be the classical Hopf manifold corresponding the diagonal $(\mu, \ldots, \mu)$. In this case, the degree of line bundles can be computed explicitly (for details in the case of surfaces, see [LT, T]); it is determined by a map from $\mathrm{Pic}(X)$ to the reals, denoted $\deg : \mathrm{Pic}(X) \to \mathbb{R}$, of the form $z \mapsto C \ln |z|$, where C is a real constant. We define the degree of the line bundle L_a, for $a \in \mathbb{C}^*$, as

$$\deg L_a = \ln |a| / \ln |\mu|,$$

with the normalisation chosen so that $\deg \pi^*(\mathcal{O}_{\mathbb{P}^n}(m)) = \deg L_{\mu^n} = m$.

Example 2.4. For a generic Hopf manifold X given by the diagonal $(\mu_1, \ldots, \mu_n)$, $0 < |\mu_1| \leq \cdots \leq |\mu_n| < 1$, we define the degree of the line bundle L_a, for $a \in \mathbb{C}^*$, as

$$\deg L_a = -\ln |a|$$

so that, given any positive integer m, we have $\deg L_{\mu_i^m} \geq 0$ for all $i = 1, \ldots, n$. In fact, we shall see in section 3.4 that if we were to define the degree using instead a positive constant C, then every filtrable rank-2 vector bundle on a generic Hopf surface would be unstable.

3. Holomorphic vector bundles on Hopf surfaces

3.1. **Topologically trivial holomorphic vector bundles.** We begin by considering topologically trivial holomorphic vector bundles of arbitrary rank on diagonal Hopf surfaces, proving that they possess filtrations by vector bundles.

Proposition 3.1. *On a Hopf surface X, topologically trivial holomorphic vector bundles of rank greater than 1 are not simple.*

Proof. We prove the proposition by contradiction. Consider a holomorphic vector bundle E on X with $c_1(E) = c_2(E) = 0$ and assume that it is simple, implying that $h^0(X; \mathrm{ad}(E)) = 0$. Recall that the canonical bundle of X is given by $\mathcal{O}_X(-D)$, where D is the effective divisor $T_1 + T_2$. The inclusion $K_X = \mathcal{O}_X(-D) \subset \mathcal{O}_X$ then gives

$$h^2(X; \mathrm{ad}(E)) = h^0(X; \mathrm{ad}(E) \otimes K_X) = h^0(X; \mathrm{ad}(E)) = 0.$$

Consequently, since $\chi(E) = 0$, we have $h^1(X; \mathrm{ad}(E) \otimes K_X) = h^1(X; \mathrm{ad}(E)) = 0$. Inserting this into the long exact sequence on cohomology associated to the exact sequence

$$0 \to \mathrm{ad}(E) \otimes K_X \to \mathrm{ad}(E) \to \mathrm{ad}(E)|_D \to 0,$$

we obtain $h^0(D; \mathrm{ad}(E)|_D) = 0$, which contradicts the fact that $h^0(D; \mathrm{ad}(E)|_D) \geq h^0(T_i; \mathrm{ad}(E)|_{T_i}) \geq 1$. □

Given that we are considering bundles on a surface, we then obtain the following.

Corollary 3.2. *Any topologically trivial holomorphic vector bundle on a Hopf surface X possesses a filtration by vector bundles.* □

Holomorphic vector bundles on Hopf manifolds that admit filtrations by vector bundles have been studied by Mall [Ma2, Ma3]. In particular, he shows that they are the only bundles that can be constructed using factors of automorphy. Rank-2 vector bundles on a generic Hopf surface X can be classified as follows.

Proposition 3.3 (Mall)**.** *Let E be an extension of line bundles on X. Then, there exists an exact sequence*

$$0 \longrightarrow L_a \longrightarrow E \longrightarrow L_b \longrightarrow 0,$$

with $a, b \in \mathbb{C}^*$. We have the following possibilities:

(i) If $a = b\mu_1^{m_1}\mu_2^{m_2}$, for non-negative integers m_1 and m_2, then $E = L_a \oplus L_b$ or E is the unique *non-trivial extension $0 \longrightarrow L_a \longrightarrow E \longrightarrow L_b \longrightarrow 0$.*

(ii) If $ab^{-1} \neq \mu_1^{m_1}\mu_2^{m_2}$ for all integers $m_1, m_2 \geq 0$, then $E = L_a \oplus L_b$.

Remark 3.4. The rank-2 vector bundles described in Proposition 3.3 are given by the factors of automorphy:

$$\begin{pmatrix} a & \epsilon z_1^{m_1} z_2^{m_2} \\ 0 & b \end{pmatrix}, \tag{3.5}$$

where $\epsilon = 0$ if the bundle is decomposable and $\epsilon = 1$ otherwise [Ma2].

3.2. **Constructing rank-2 vector bundles.** There are three standard ways for constructing rank-2 vector bundles on a surface X.

(i) Double covers. One method for constructing rank-2 vector bundles on a surface X is the following. Find a smooth double cover $\varphi : Y \to X$ of X. Then, for any line bundle L on Y, the direct image $\varphi_* L$ is a rank-2 vector bundle.

(ii) Serre construction. This method consists in finding locally free extensions of the form

$$0 \longrightarrow L \longrightarrow E \longrightarrow L' \otimes I_Z \longrightarrow 0,$$

where L and L' are line bundles on X and I_Z is the ideal sheaf Z of a finite set of points (counting multiplicity) on X that may be empty. Bundles thus obtained are clearly all *filtrable*.

Note that if X is a Hopf surface, then $c_2(E) = l(Z)$, that is, it is equal to the number of points in Z (counting multiplicity). In addition, extensions of this type exist for any choice of line bundles L and L' and points on X.

Remark. Every rank-2 vector bundle on an algebraic surface is filtrable and can be realised via the Serre construction. On non-algebraic surfaces, there exist, however, *non-filtrable* bundles. In fact, we shall see that rank-2 vector bundles on Hopf surfaces are generically non-filtrable.

(iii) Elementary modifications. Start with a rank-2 vector bundle E, an effective divisor D on X, and a line bundle λ on D such that the restriction of E to D admits a projection $p : E|_D \to \lambda$. Let $i : D \to X$ denote the natural inclusion. Then, if we also denote by p the induced projection $E \to i_*\lambda$ on X, where $i_*\lambda$ is now a torsion sheaf supported on D, we have the following exact sequence on X:

$$0 \longrightarrow \bar{E} \longrightarrow E \xrightarrow{p} i_*\lambda \longrightarrow 0,$$

where $\bar{E} := \ker p$ is a rank-2 locally free sheaf; it is called the *elementary modification of E induced by p*. Note that $\bar{E}$ is isomorphic to E away from D. In addition, $\det(\bar{E}) = \mathcal{O}_X(-D) \otimes \det E$ and, on a Hopf surface, we have $c_2(\bar{E}) = c_1(L) + c_2(E)$. Other properties of elementary modifications can be found, for example, in [F]

Remark. We shall see that on elliptically fibred Hopf surfaces, all rank-2 vector bundles can be obtained this way, but that these methods only produce filtrable vector bundles on generic Hopf surfaces. In the latter case, the only method we know so far for constructing non-filtrable vector bundles is to choose vector bundles on the open cover X_1, X_2 of X that are isomorphic on the overlap and gluing them to obtain a vector bundle on X. Unfortunately, this method makes their classification very difficult.

3.3. **Notation and terminology.** To study bundles on a Hopf surface X, one of our main tools is restriction to one of its elliptic curves $T = \mathbb{C}^*/\mu$. It is important to point out that the restriction of any vector bundle E on X to T is generically semistable, given by an extension of line bundles of degree zero; if these line bundles correspond to the factors of automorphy a and b in $\mathbb{C}^*/\mu$, we say that E has *splitting type* (a, b) on T. In fact, the restriction of a vector bundle is unstable on at most an isolated set of curves on X. If the restriction of E to T is unstable, we say that the vector bundle has a *jump* over T.

Consider a rank-2 vector bundle E on X with determinant δ that has a jump of multiplicity m over the curve T. The restriction of E to T is then of the form $\lambda \oplus (\lambda^* \otimes \delta_{x_0})$, for some $\lambda \in \mathrm{Pic}^{-h}(T)$, $h > 0$; the integer h is called the *height* of the jump at T. Moreover, up to a multiple of the identity, there is a *unique* surjection $E|_T \to \lambda$, which defines a canonical elementary modification of E that we denote $\bar{E}$; this elementary modification is called *allowable* [F]. Therefore, we can associate to E a finite sequence $\{\bar{E}_1, \bar{E}_2, \ldots, \bar{E}_l\}$ of allowable elementary modifications such that $\bar{E}_l$ is the only element of the sequence that does not have a jump at T. The integer l is called the *length* of the jump at T.

Note. Note that if a vector bundle E jumps over the curve T with mulplicity m, then $m = \sum_{i=1}^{l-1} h_i$, where $h_0 = h$ is the height of E and h_i is the height of $\bar{E}_i$, $i = 0, \ldots, l-1$. Moreover, if E has k jumps of multiplicity $m_1, \ldots, m_k$, respectively, then $\sum_{i=1}^{k} m_i = c_2(E)$. For a detailed description of jumps, we refer the reader to [Mo1, BrMo3].

3.4. **Moduli spaces.** For a fixed line bundle L_δ on X, let $\mathcal{M}_{\delta,c_2}$ be the moduli space of stable holomorphic rank-2 vector bundles with determinant L_δ and second Chern class c_2. Referring to Proposition 3.1, the moduli space $\mathcal{M}_{\delta,c_2}$ is non-empty if only if $c_2 > 0$, in which case, a similar computation shows that it is a complex manifold of dimension $4c_2$.

Stable bundles on elliptic Hopf surfaces have been described in detail in [Mo1, BrMo3]. Consequently, we focus our presentation on the generic case, briefly stating

the results for classical Hopf surfaces that will be needed in section 3.5. Recall that non-filtrable bundles are automatically stable. We therefore only determine stability conditions for the filtrable ones.

3.4.1. *Filtrable rank-2 vector bundles.* Consider a filtrable rank-2 vector bundle E on X with $c_2(E) = c_2 > 0$. It can therefore be expressed as an extension of the form:

$$0 \longrightarrow L_a \longrightarrow E \longrightarrow L_b \otimes I_Z \longrightarrow 0,$$

where Z is a set of c_2 points (counting multiplicity). On a classical Hopf surface, every filtrable bundle can also be constructed by starting with a bundle with trivial Chern class and adding jumps to it to obtain a bundle with the desired second Chern class and ideal sheaf I_Z. This is done by performing elementary modifications, using the fibres of the elliptic fibration π that contain the points of Z. In particular, this is always possible because every point on the surface lies on an elliptic curve. The advantage of thinking of filtrable bundles this way is that it enables us to completely classify them. Unfortunatly, since generic Hopf surfaces possess only two curves, many filtrable vector bundles cannot be constructed this way. We nevertheless have a good description of stable filtrable rank-2 vector bundles; necessary and sufficient conditions for stability can be stated as follows.

Theorem 3.6. *Consider a filtrable rank-2 vector bundle E on a generic Hopf surface that has determinant δ and jumps on T_1 and T_2 of lengths l_1 and l_2, respectively. Suppose that L_a is one of the line bundles of maximal degree mapping into E. Then, E is stable if and only if*

$$a^2 = \delta\mu_1^{-l_1-k_1}\mu_2^{-l_2-k_2} \tag{3.7}$$

for two non-negative integers k_1 and k_2 that are not both zero, or

$$a \in D_{l_1,l_2} := \left\{\alpha \in \mathbb{C}^* \ : \ |\delta|^{1/2} < |\alpha| < |\delta\mu_1^{-2l_1}\mu_2^{-2l_2}|^{1/2}\right\}.$$

In particular, this implies that $l_1 + l_2 > 0$ unless a satisfies equation (3.7).

Proof. Recall that E is stable if and only if each of its destabilising bundles has degree less than $\deg(L_\delta)/2$. Consider the rank-2 vector bundle $\bar{E}$ obtained by performing l elementary modifications to remove the jumps of E. Then, $\det \bar{E} = L_{\delta\mu_1^{-l_1}\mu_2^{-l_2}}$. Note that E and $\bar{E}$ have the same destabilising bundles. Indeed, $\bar{E}$ is obtained by taking consecutive elementary modifications determined by exact sequences of the form:

$$0 \to \bar{E}_{i+1} \to \bar{E}_i \to i_*\lambda \to 0,$$

where λ is a line bundle of negative degree on T_1 or T_2, for $0 \le i \le l_s - 1$, $s = 1, 2$. Thus, $h^0(X, L_{c^{-1}} \otimes i_*\lambda) = 0$ for all line bundles L_c, so that

$$h^0(X, L_{c^{-1}} \otimes \bar{E}_{i+1}) = h^0(X, L_{c^{-1}} \otimes \bar{E}_i).$$

This implies, in particular, that $\bar{E}$ is an extension of $L_{a^{-1}\delta\mu_1^{-l_1}\mu_2^{-l_2}} \otimes I_{Z'}$ by L_a, where Z' is a (possibly empty) set of points that do not lie on T_1 or T_2.

We therefore have to determine the line bundles that map non-trivially into $\bar{E}$. Let L_c be such a line bundle. We first assume that Z' is empty. Suppose that $\bar{E}$ decomposes as $L_a \oplus L_{a^{-1}\delta\mu_1^{-l_1}\mu_2^{-l_2}}$; then L_a and $L_{a^{-1}\delta\mu_1^{-l_1}\mu_2^{-l_2}}$ are the line bundles of maximal degree mapping into $\bar{E}$. To ensure the stability of E, both must have

degree strictly smaller than $\deg(L_\delta)/2$. Referring to the definition of degree given in section 2.7, this is equivalent to a being an element of D_{l_1,l_2}.

If $\bar{E}$ is instead indecomposable, then there exist $a \in \mathbb{C}^*$ and non-negative integers m_1, m_2 such that $\bar{E}$ is given by the non-trivial extension of $L_{a\mu_1^{-m_1}\mu_2^{-m_2}}$ by L_a (see Proposition 3.3 (i)). Note that any line bundle mapping into $\bar{E}$ must also map to L_a. Indeed, if $h^0(X, L_{c^{-1}}\bar{E}) \neq 0$, then $h^0(X, L_{c^{-1}a})$ and $h^0(X, L_{c^{-1}a\mu_1^{-m_1}\mu_2^{-m_2}})$ cannot both zero; hence, since $h^0(X, L_{c^{-1}a\mu_1^{-m_1}\mu_2^{-m_2}}) \leq h^0(X, L_{c^{-1}a})$, we have $h^0(X, L_{c^{-1}a}) \neq 0$ and $c^{-1}a = \mu_1^{k_1}\mu_2^{k_2}$ for some integers $k_1, k_2 \geq 0$. Consequently, L_a is the (unique) line bundle of maximal degree mapping into $\bar{E}$. Suppose that it has degree strictly smaller than $\deg(L_\delta)/2$. Then, since

$$a^2\mu_1^{-m_1}\mu_2^{-m_2} = \delta\mu_1^{-l_1}\mu_2^{-l_2}$$

and $|\mu_1^{-m_1}\mu_2^{-m_2}| > 1$, it follows that a is an element of D_{l_1,l_2}.

Let us now assume that Z' is not empty. Note that $\bar{E}$ is semistable on both T_1 and T_2. To simplify the notation, let us set $b = a^{-1}\delta\mu_1^{-l_1}\mu_2^{-l_2}$. The set of line bundles mapping into $\bar{E}$ is therefore contained in

$$\left\{L_c \ : \ c = a\mu_1^{-k_1}\mu_2^{-k_2} \quad \text{or} \quad c = b\mu_1^{-k_1}\mu_2^{-k_2} \text{ for integers } k_1, k_2 \geq 0\right\}.$$

Let L_c be another destabilising line bundle of $\bar{E}$ so that E/L_c is torsion free. Then $c = b\mu_1^{-k_1}\mu_2^{-k_2}$ for integers $k_1, k_2 \geq 0$. If at least one of the integers is non-zero, then we must have $c = a$, otherwise the quotient E/L_c is not torsion-free (recall that on a generic Hopf surface, there are no non-trivial relations between μ_1 and μ_2). Consequently, L_a is the (only) line bundle of maximal degree mapping into E; since $a = a^{-1}\delta\mu_1^{-l_1-k_1}\mu_2^{-l_2-k_2}$ with k_1 or k_2 non-zero, we have $|a| > |\delta|^{1/2}$, implying that E is stable. Finally, if $k_1 = k_2 = 0$, then the second destabilising line bundle of E is $L_{a^{-1}\delta\mu_1^{-l_1}\mu_2^{-l_2}}$ so that E is stable if and only if $a \in D_{l_1,l_2}$. □

Remark 3.8. On a classical Hopf surface X, determined by a diagonal (μ, μ), vector bundles can have jumps on elliptic curves other than T_1 and T_2. Let E be a filtrable rank-2 vector bundle on X that has determinant L_δ and k jumps of lengths $l_1, \ldots, l_k$, respectively. Set $l = l_1 + \cdots + l_k$. Using the notation of Theorem 3.6, an extension of $L_{a^{-1}\delta} \otimes I_Z$ by L_a is stable if and only if $a \in D_l$, where

$$D_l := \left\{\alpha \in \mathbb{C}^* \ : \ |\delta|^{1/2} < |\alpha| < |\delta\mu^{-2l}|^{1/2}\right\}.$$

This follows from Lemma 4.5 and Corollary 4.6 of [Mo1] (with $\mu = \lambda^{-1}$ because classical Hopf surfaces were defined by a complex number λ with $|\lambda| > 1$ in [Mo1]).

Remark 3.9. The domains D_{l_1,l_2} and D_l defined in Theorem 3.6 and Remark 3.8 are independent of the definition of degree, up to multiplication by a positive constant. Otherwise, one readily verifies that these domains would in fact be empty.

We now describe stable filtrable bundles with $c_2 = 1$ and fixed determinant δ that have a jump on T_1 or T_2. Without loss of generality, we assume it to be T_1. Note that a similar analysis can be carried out for bundles with $c_2 > 1$.

Proposition 3.10. *Let E be a stable filtrable rank-2 vector bundle on X with determinant δ and a jump of multiplicity 1 on T_1. Then, E is uniquely determined by a triple (a, λ, p) such that*

$$(a, \lambda) \in D_{1,0} \times \mathrm{Pic}^1(T_1)$$

and p is a projection from $L_{a^{-1}\delta} \oplus L_{a\mu_1}$ to λ on T_1 that is unique up to isomorphism, unless $a^2 = \delta\mu_1^{m_1-1}\mu_2^{m_2}$ with $m_1 \geq 1$ and $m_2 > 0$, in which case it is an element of the projective line $\mathbb{P}^1(H^0(T_1, Hom(L_{a^{-1}\delta} \oplus L_{a\mu_1}, \lambda)))$.

Note that L_a corresponds to one of the destabilising line bundles of E and that λ is such that the restriction of E to T_1 splits as $\lambda \oplus (\lambda^{-1} \otimes \delta)$.

Proof. Let E be a stable filtrable rank-2 vector bundle on X with a jump of multiplicity 1 on T_1. Such a bundle E is then given by an extension of the form

$$0 \to L_a \to E \to L_{a^{-1}\delta} \otimes I_p,$$

where $a \in D_{1,0}$ and p is a point on T_1. Moreover, the allowable elementary modification $\bar{E}$ of E is an extension of $L_{a^{-1}\delta\mu_1^{-1}}$ by L_a that splits unless $a^2 = \delta\mu_1^{m_1-1}\mu_2^{m_2}$ with $m_1 \geq 1$ and $m_2 > 0$ (note that if $m_2 = 0$, then $a \notin D_{1,0}$). In the latter case, $\bar{E}$ is given by a factor of automorphy of the form (3.5) with $\epsilon = 0$ or 1.

Suppose that the splitting type of E on T_1 is $\lambda \oplus (\lambda^{-1} \otimes \delta)$ for some $\lambda \in \mathrm{Pic}^1(T_1)$. Then, E can be recovered from $\bar{E}$ by using the line bundle λ to introduce a jump to $\bar{E} \otimes L_{\mu_1}$ on T_1. Given a fixed choice of line bundle λ in $\mathrm{Pic}^1(T_1)$, we therefore have to determine which projections p from $\bar{E} \otimes L_{\mu_1}|_{T_1}$ to λ induce isomorphic elementary modifications. Note that since no element a in $D_{1,0}$ is such that $a^2 \equiv \delta \mod \mu_1$, such projections exist for all extension of $L_{a^{-1}\delta}$ by $L_{a\mu_1}$. In addition, any two such projections differ an element of $Aut(\bar{E} \otimes L_{\mu_1}|_{T_1})$ because $\deg \lambda = 1$; it is therefore sufficient to find out which automorphisms of $\bar{E} \otimes L_{\mu_1}|_{T_1}$ extend to automorphisms of $\bar{E} \otimes L_{\mu_1}$ on X.

If $\bar{E} \otimes L_{\mu_1}$ is decomposable, then $Aut(\bar{E} \otimes L_{\mu_1}|_{T_1}) = Aut(\bar{E} \otimes L_{\mu_1})$; in this case, there is a unique way, up to isomorphism, of introducing the jump so that E is uniquely determined by the pair (a, λ). Let us now assume that $\bar{E} \otimes L_{\mu_1}$ is a non-trivial extension of $L_{a^{-1}\delta}$ by $L_{a\mu_1}$, where $a^2 = \delta\mu_1^{m_1-1}\mu_2^{m_2}$ with $m_1 \geq 1$ and $m_2 > 0$. In this case, the restriction of $\bar{E} \otimes L_{\mu_1}$ to T_1 is decomposable and the only elements of $Aut(\bar{E} \otimes L_{\mu_1}|_{T_1})$ that extend to $Aut(\bar{E} \otimes L_{\mu_1})$ are multiples of the identity. Consequently, the vector bundle E is determined by a triple (a, λ, p) such that $a \in D_{1,0}$, $\lambda \in \mathrm{Pic}^1(T_1)$, and p is a projection in $\mathbb{P}^1(H^0(T_1, Hom(L_{a^{-1}\delta} \oplus L_{a\mu_1}, \lambda)))$. □

Remark 3.11. Note that if $|\mu_1| = |\mu_2|$, then the description simplifies. In this case, the allowable elementary modification $\bar{E}$ always decomposes as $L_a \oplus L_{a^{-1}\delta\mu_1^{-1}}$, otherwise $a \notin D_{1,0}$; stable filtrable bundles with $c_2 = 1$, determinant δ, and a jump on T_1 are therefore in one-to-one correspondence with the pairs in $D_{1,0} \times \mathrm{Pic}^1(T_1)$. In addition, if we consider such bundles on a classical Hopf surface $X = \mathbb{C}^2_*/(\mu, \mu)$, then we obtain the same description, except that the bundles can now have a jump over any fibre of the elliptic fibration. For a fixed fibre T, these bundles are parametrised by $D_1 \times \mathrm{Pic}^1(T)$.

As direct consequence of the above discussion, we have:

Corollary 3.12. *Consider the moduli space $\mathcal{M}_{\delta,1}$ of stable rank-2 vector bundles on a Hopf surface X with determinant δ and second Chern class 1. Let $F\mathcal{M}_{\delta,1}$ be the subset of $\mathcal{M}_{\delta,1}$ consisting of filtrable bundles. Then every component of $F\mathcal{M}_{\delta,1}$ has codimension at least one in $\mathcal{M}_{\delta,1}$.* □

Example 3.13. We end this section with an application of the above analysis to magnetic monopoles on solid tori. These can be seen to correspond to S^1-invariant

instantons on Hopf surfaces of the form

$$\mathbb{C}^2_*/(\mu, |\mu|),$$

where μ is a complex number with $|\mu| < 1$ [B1, B2]. Monopoles on solid tori can therefore also be identified, via the Hitchin-Kobayashi correspondence [LT], with $\mathbb{C}^*$-equivariant stable holomorphic bundles on Hopf surfaces.

Moduli spaces of $\mathbb{C}^*$-equivariant stable holomorphic bundles on Hopf surfaces were first studied by Braam and Hurtubise [BH] in the classical case. They showed that the moduli spaces $\mathcal{M}(m,k)$ of monopoles of mass m and charge k are complex manifolds of dimension $2k$ consisting of certain stable filtrable rank-2 vector bundles with $c_2 = mk$ and a jump of height k and length m over T_1. In particular, they showed that $\mathcal{M}(m,1)$ is isomorphic $D_m \times \mathrm{Pic}^1(T_1)$, by classifying $\mathbb{C}^*$-equivariant bundles on X_1 and X_2 and determining how one can glue them on the overlap to obtain distinct monopoles. The method presented in Proposition 3.10 offers, however, a more invariant way of approaching the problem. In fact, a similar analysis shows that for $k > 1$, the moduli space $\mathcal{M}(m,k)$ consists of triples (a, λ, p), where $a \in D_m$, $\lambda \in \mathrm{Pic}^k(T_1)$, and p is a projection on T_1 from $L_{a\mu^m} \oplus L_{a^{-1}\delta}$ to λ or possibly from the non-trivial extension of $L_{a^{-1}\delta}$ by $L_{a\mu^m}$ to λ when $a^2 \equiv \delta^{-1}$ mod μ. The space of such projections is in this case $(2k-2)$-dimensional.

3.4.2. *Non-Filtrable bundles.* In this section, we turn to the problem of constructing non-filtrable holomorphic vector bundles. We begin by noting that there exist many non-filtrable bundles on Hopf surfaces. For example, the generic elements of $\mathcal{M}_{\delta,1}$ are non-filtrable because the set of filtrable bundles in $\mathcal{M}_{\delta,1}$ has codimension at least one (see sections 3.2 and 3.4.1). Such vector bundles can then be used to construct non-filtrable vector bundles of arbitrary second Chern class by performing elementary modifications to add (or increase) jumps.

On elliptically fibred Hopf surfaces, non-filtrable holomorphic vector bundles are by now well-understood. In fact, one can show that they can all be constructed by using double covers and elementary modifications (for details, see [Mo1, BrMo1, BrMo2]). For the convenience of the reader, we briefly recall how this is done in the case of a classical Hopf surface.

Let X be a classical Hopf surface. Referring to section 2, it admits an elliptic fibration $\pi : X \to \mathbb{P}^1$, with fibre an elliptic curve T and relative Jacobian $J(X) = \mathbb{P}^1 \times T^*$. Consider a vector bundle E on X. One of the main tools for studying this bundle is restriction to the fibres of the fibration π. In particular, there exists a divisor S_E in the relative Jacobian of X, called the *spectral curve* or *cover* of the bundle, that encodes the isomorphism class of the bundle E over each fibre of π. Note that the self-intersection of this divisor is equal to a multiple of the second Chern class of the bundle.

Example. Let E be a rank-2 vector bundle on X with determinant L_δ and second Chern class c_2. Then it has a spectral curve of the form

$$S_E := \left(\sum_{i=1}^{k} \{x_i\} \times T^* \right) + C_E,$$

where C_E is a bisection of $J(X)$ and $x_1, \cdots, x_k$ are points in $\mathbb{P}^1$ corresponding to the jumps of E. Away from the jumps, the pair of points $(\lambda_0, \lambda_0^{-1} \otimes L_\delta)$ on C_E above $x_0 \in \mathbb{P}^1$ gives the splitting type of E on the fibre $T_{x_0} = \pi^{-1}(x_0)$. In this

case,

$$S_E \cdot S_E = 4c_2.$$

Moreover, if the bisection C_E is smooth, then it is a double cover of $\mathbb{P}^1$ of genus $(2c_2 - 2k - 1)$.

Let us now describe non-filtrable vector bundles on X. We have seen that we can associate to any rank-2 vector bundle E on X a bisection $C_E \subset J(X)$. The vector bundle E is then filtrable if and only if its bisection C_E is reducible. Conversely, given any bisection C of $J(X)$, one can associate to it at least one rank-2 vector bundle on X. This implies, in particular, that non-filtrable bundles exist on classical Hopf surfaces because irreducible bisections exist in $J(X)$. The bundles determined by C are constructed as follows. Consider the double cover

$$\varphi : Y := X \times_{\mathbb{P}^1} C \to X;$$

then, for any line bundle L on Y, the pushdown $\varphi_* L$ is a rank-2 vector bundle on X. In fact, for a certain class of line bundles on Y, the resulting rank-2 vector bundles will have spectral cover C. For example, if the bisection is smooth, then one can show that the bundles that correspond to it are parametrised by the abelian variety $Jac(C)$ (see [Mo1] for precise statements).

Note. The spectral construction applies, in fact, to any elliptic fibration; it has been used by many authors to study bundles on elliptic fibrations (see for example [F, FM, FMW, D, T]).

Any holomorphic rank-2 vector bundle on a classical Hopf surface can therefore be constructed by using a double cover and elementary modifications (to add jumps). This is, however, certainly not the case for generic Hopf surfaces.

Proposition 3.14. *On a generic Hopf surface $X = \mathbb{C}^2_*/(\mu_1, \mu_2)$, only filtrable vector bundles can be constructed by using double covers.*

Proof. Consider a double cover $\varphi : Y \to X$ of X determined by the line bundle $\mathcal{L}$ on X. Then, for any line bundle $\mathcal{M}$ on X, we have

$$c_2(\varphi_*(\mathcal{M}) = \frac{1}{2}(c_1^2(\mathrm{Nm}\mathcal{M})) - \varphi_*(c_1^2(\mathcal{M})) - \varphi_*(c_1(\mathcal{M})).c_1(\mathcal{L}))$$

(see [Br]). Therefore, since $h^2(X, \mathbb{Z}) = 0$, this reduces to

$$c_2(\varphi_*(\mathcal{M}) = -\varphi_*(c_1^2(\mathcal{M})). \tag{3.15}$$

Without loss of generality, we can assume that Y is a smooth surface (otherwise, take its normalisation). Referring to Lemma 2.1, Y is then either a generic Hopf surface or one of the non-primary Hopf surfaces $\Theta^*_{-2}/(\beta, \mu_1)$, with $\beta^2 = \mu_1$, described in section 2.4. This means, in particular, that the first Chern class of any line bundle L on X is torsion. Consequently, by (3.15), we have $c_2(\varphi_*(L)) = 0$. The rank-2 vector bundle $\varphi_* L$ is thus filtrable by Corollary 3.2. □

3.5. **Poisson structures.** A (holomorphic) Poisson structure on a complex surface is given by a global section of its anticanonical bundle [Bo]. Consequently, any Hopf surface X admits a Poisson structure because its anticanonical bundle K_X^{-1} is given by the effective divisor $D := T_1 + T_2$. Fix a Poisson structure $s \in H^0(X, K_X^{-1})$ on X. A Poisson structure $\theta = \theta_s \in H^0(\mathcal{M}, \otimes^2 T\mathcal{M})$ on the moduli space $\mathcal{M} := \mathcal{M}_{\delta,c_2}$

is then defined as follows: for any bundle $E \in \mathcal{M}$, $\theta(E) : T^*_E\mathcal{M} \times T^*_E\mathcal{M} \longrightarrow \mathbb{C}$ is the composition

$$\theta(E) : H^1(X, \mathrm{ad}(E) \otimes K_X) \times H^1(X, \mathrm{ad}(E) \otimes K_X) \xrightarrow{\circ} H^2(X, \mathrm{End}(E) \otimes K_X^2) \xrightarrow{s} H^2(X, \mathrm{End}(E) \otimes K_X) \xrightarrow{\mathrm{Tr}} \mathbb{C},$$

where the first map is the cup-product of two cohomology classes, the second is multiplication by s, and the third is the trace map.

The Poisson structure s is degenerate, its divisor being $D = (s)$. Moreover, at any point $E \in \mathcal{M}$,

$$\mathrm{rk}\,\theta(E) = 4c_2 - \dim H^0(D, \mathrm{ad}(E|_D)).$$

We see that the rank of the Poisson structure is generically $4c_2 - 2$, and "drops" at the points of $\mathcal{M}$ corresponding to bundles that are not regular over the fibres T_1 and T_2 (for details in the elliptic case, see [Mo1]).

On the set of bundles for which the Poisson structure is maximal, one can define the following maps:

$$\begin{array}{rcl} f_i : \mathcal{M}_{\delta,c_2} & \longrightarrow & \mathrm{Pic}^0(T_i)/i_\delta = \mathbb{P}^1 \\ E & \mapsto & (L_{a_i}|_{T_i}, L_{a_i^{-1}\delta}|_{T_i}), \end{array}$$

where L_{a_i} and $L_{a_i^{-1}\delta}$ are the destabilising bundles of $E|_{T_i}$, and i_δ is the involution of $\mathrm{Pic}^0(T_i)$ given by $\lambda_0 \mapsto \lambda_0^{-1} \otimes L_\delta$. The functions f_1 and f_2 are linearly independent Casimirs; this can be proven as in the elliptic case [Mo1].

On a classical Hopf surface X, the Casimirs can be described very explicitly in terms of the spectral data of the bundle. Let E be rank-2 vector bundle on X with determinant L_δ and spectral curve $S_E \subset J(X)$ (see section 3.4.2). In this case, one can associated to E an equivalent divisor that is constructed as follows. Consider the quotient of $J(X) = \mathbb{P}^1 \times T^*$ by the involution $i_\delta := id \times i_\delta$, where id is the identity on $\mathbb{P}^1$ and i_δ is the involution of T^* determined by L_δ (see above). Let $\eta : J(X) \to J(X)/i_\delta = \mathbb{P}^1 \times \mathbb{P}^1$ be the canonical map. By construction, the spectral curve of E is invariant with respect to the involution i_δ. It therefore descends to a divisor on $\mathbb{P}^1 \times \mathbb{P}^1$ of the form

$$\mathcal{G}_E := \left(\sum_{i=1}^{k} \{x_i\} \times \mathbb{P}^1 \right) + \mathrm{Gr}(F_E),$$

where $\mathrm{Gr}(F_E)$ is a the graph of a rational map $F_E : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $(c_2(E) - k)$ such that $\eta^*\mathrm{Gr}(F_E) = C_E$. This divisor is called the *graph* of E; it is an element of the linear system $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(c_2, 1)|$. Note that the bundle E is filtrable if and only if the map F_E is constant.

Let x_1 and x_2 be the points in $\mathbb{P}^1$ such that $\pi^{-1}(x_i) = T_i$, for $i = 1, 2$. The Casimirs $f_i : \mathcal{M}_{\delta,c_2} \to \mathbb{P}^1$, $i = 1, 2$, are then given by $E \mapsto F_E(x_i)$, where F_E is the graph of E. These functions can be used to describe the symplectic leaves of the Poisson structure; the symplectic leaves of maximal rank are labelled as follows:

$$\mathcal{L}_{C_1,C_1} := \{E \in \mathcal{M}_{\delta,c_2} \mid \mathrm{rk}\,\theta(E) = 4n - 2 \text{ and } f_i(E) = C_i \text{ for } i = 1, 2\}.$$

On the open dense subset of non-filtrable bundle that are regular on every fibre of π, the elements of the leaf $\mathcal{L}_{C_1,C_1}$ can be identified with pairs of the form $(Gr(F), E)$, where $Gr(F)$ is the graph of a rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$ of degree c_2 passing through the points (x_1, C_1) and (x_2, C_2) in $\mathbb{P}^1 \times \mathbb{P}^1$ and E is a rank-2 vector bundle

whose graph is given by $Gr(F)$. Recall that for such a graph, the set of all bundles corresponding to it is parametrised by the Jacobian of a curve of genus $2c_2 - 1$, given by $C = \eta^* Gr(F)$, confirming that the leaf $\mathcal{L}_{C_1,C_1}$ is $(4c_2 - 2)$-dimensional.

It is in fact possible to give an explicit description of all the symplectic leaves of the Poisson structure. We finish by describing these leaves for the moduli space $\mathcal{M}_{\delta,1}$ on a classical Hopf surface.

Example 3.16. Let us first consider stable vector bundles E in $\mathcal{M}_{\delta,1}$ where the Poisson structure θ has maximal rank 2. On a classical Hopf surface, $f_1(E) = f_2(E)$ if and only if the vector bundle E filtrable (because if the map F_E has degree one, then it must be injective). Let us assume that $C_1 = C_2$; then,

$$\mathcal{L}_{C_1,C_1} = \{(x_0, \lambda) \mid x_0 \in \mathbb{P}^1 \backslash \{x_1, x_2\} \text{ and } \lambda \in \mathrm{Pic}^1(T_{x_0})\}.$$

Indeed, stable filtrable bundles on elliptic Hopf surfaces with $c_2 = 1$ are completely determined by the choice of a fibre $T_{x_0} = \pi^{-1}(x_0)$, over which they have a jump, and a pair (a, λ) in $D_1 \times \mathrm{Pic}^1(T_{x_0})$ (see Remark 3.11). However, the Casimirs fix a so that x_0 and λ are the only free parameters left. If $C_1 \neq C_2$, then the leaf $\mathcal{L}_{C_1,C_1}$ is given as above.

Finally, at the remaining points of the moduli space, the Poisson structure has rank zero. These points correspond to stable filtrable bundles that have a jump of multiplicity one on either T_1 or T_2. Referring to Remark 3.11, these bundles are completely determined by their restrictions to the curves T_1 or T_2, which fix the pairs (a, λ) parameterising them.

Remark 3.17. If X is a non-generic Hopf surface, then the graph map

$$G : \mathcal{M}_{\delta,c_2} \to |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(c_2, 1)| = \mathbb{P}^{2c_2+1},$$

which associates to each bundle its graph, admits a structure of algebraically completely integrable Hamiltonian systems [Mo1, BrMo3].

4. Holomorphic vector bundles on higher dimensional Hopf manifolds

We have seen that holomorphic vector bundles on Hopf surfaces are generically non-filtrable. In contrast, they are always filtrable on higher dimensional Hopf manifolds. More precisely, the following is known in this case [V1, V2].

Theorem 4.1 (Verbitsky)**.** *Let X be a diagonal Hopf manifold of dimension greater than two. Then all coherent sheaves on X are filtrable.*

Theorem 4.2 (Verbitsky)**.** *Let $\pi : X \to \mathbb{P}^{n-1}$ be a classical Hopf manifold of dimension $n \geq 3$. Let E be a stable holomorphic vector bundle E on X. Then $E = L \otimes \pi^* E'$, where L is a line bundle on X and E' is a stable bundle on $\mathbb{P}^{n-1}$.*

These results were first proven for positive principal elliptic fibrations of dimension greater than two [V1], examples of which are given by classical Hopf manifolds. Verbitsky shows that on such an elliptic fibration $\pi : X \to M$, where M is a Kähler manifold dimension at least two, stable vector bundles are equivariant with respect to a torus action. He then uses this to show that stable bundles on X are of form the $L \otimes \pi^* E'$; when the base M is projective, this implies that all holomorphic vector bundles are filtrable. His equivariance argument extends, however, to give filtrability of bundles on all diagonal Hopf manifolds [V2].

Note that on a classical Hopf manifold X, the filtrability of vector bundles can easily be seen via the spectral construction presented in section 3.4.2 because bundles on such a manifold are topologically trivial.

Proposition 4.3. *Let E be a rank-r vector bundle on X. Then E is topologically trivial. Moreover, its spectral cover S_E is an effective divisor in $J(X) = \mathbb{P}^{n-1} \times T^*$ of the form*

$$S_E = \sum_{i=1}^{r} \mathbb{P}^{n-1} \times \{\lambda_i\}, \tag{4.4}$$

for some points $\lambda_1, \ldots, \lambda_r$ in T^*, implying that it is filtrable.

Proof. Let L be a line bundle on X such that $h^0(\pi^{-1}(x), L^* \otimes E) = 0$ for generic $x \in \mathbb{P}^{n-1}$. Then, $R^1\pi_*(L^* \otimes E)$ is a torsion sheaf on $\mathbb{P}^{n-1}$ and $R^i\pi_*(L^* \otimes E) = 0$ for $i = 0$ and $i \geq 2$. Suppose that the line bundle L corresponds to the section $X \times \{\lambda\}$ of $J(X)$. Note that

$$\mathrm{Pic}(J(X)) = \mathrm{Pic}(\mathbb{P}^{n-1}) \times \mathrm{Pic}(T^*).$$

Recall that the spectral cover S_E of E is an effective divisor on $J(X)$ that is an r-to-one cover of $\mathbb{P}^{n-1}$. Consequently, we have

$$S_E \sim \sum_{i=1}^{r} \mathbb{P}^{n-1} \times \{\lambda_i\} + m\{H\} \times T^*$$

for some non-negative integer m, where the λ_i's are points in T^*, H is a hyperplane in $\mathbb{P}^{n-1}$. Then, $S_E \cdot (X \times \{\lambda\}) = m\{H\} \times \{\lambda\}$ and the support of $R^1\pi_*(L^* \otimes E)$ is a divisor on $\mathbb{P}^{n-1}$ equivalent to mH. Furthermore, if h is the Poincaré dual of H in $H^2(\mathbb{P}^{n-1}, \mathbb{Z})$, then $c_1(R^1\pi_*(L^* \otimes E)) = mh$. Note that h is the positive generator of $H^2(\mathbb{P}^{n-1}, \mathbb{Z})$. Given that $ch(E) = r + (-1)^{(n-1)}c_n(E)/(n-1)!$ and $td(X) = 1$, by Grothendieck-Riemann-Roch, it follows that

$$ch(R^1\pi_*(L^* \otimes E)) = \pi_*\left(ch(E) \cdot td(X)\right) \cdot td(\mathbb{P}^n)^{-1} = (-1)^n \frac{c_n(E)}{(n-1)!} h^{n-1}.$$

Consequently, since $n > 2$, we see that $m = c_n(E) = 0$ and S_E is of desired the form. □

A vector bundle E on a classical Hopf manifold X therefore either admits a filtration by vector bundles, which decomposes if all the λ_i's appearing in its spectral cover S_E (4.4) are distinct, or it can also be of the form $L \otimes \pi^* E'$ when all the λ_i's are equal. Unfortunatly, this analysis cannot be successfully carried out for bundles on generic Hopf manifolds as there is no satisfactory analogue of the spectral construction in the generic case (because the are so few divisors on these manifolds).

On elliptically fibred Hopf manifolds of dimension greater than two, the study of stable vector bundles boils down to the difficult problem of classifying stable vector bundles on projective spaces of dimension greater than one. On generic Hopf manifolds of dimension greater than two, the question is, however, greatly simplified by the fact that these manifolds possess few subvarieties. One can therefore obtain a complete classification of vector bundles by studying extensions of sheaves. For brevity, we restrict our presentation to the case of holomorphic rank-2 vector bundles; similar results however hold for bundles of arbitrary rank. In particular, we show that there exist stable rank-2 vector bundles on these manifolds.

Proposition 4.5. *Let X be a generic Hopf manifold of dimension $n \geq 3$ given by the quotient $\mathbb{C}^n_*/(\mu_1, \ldots, \mu_n)$. If E be a rank-2 vector bundle on X, then it is of one of the following three types:*

(i) E is decomposable and given by $L_a \oplus L_b$.

(ii) E is not decomposable and an extension of line bundle; in this case, we can write E as $L_a \otimes E'$, where E' is a non-trivial extension of the form:

$$0 \to L_{\mu_1^{m_1} \ldots \mu_n^{m_n}} \to E' \to \mathcal{O} \to 0,$$

with $m_1, \ldots, m_n$ non-negative integers.

(iii) E is not an extension of line bundle; in this case, we can write E as $L_a \otimes E'$, where E' is the unique *locally-free extension of the form:*

$$0 \to L_{\mu_1^{m_1} \ldots \mu_i^{-k_i} \ldots \mu_j^{-k_j} \ldots \mu_n^{m_n}} \to E' \to I_{k_i k_j} \to 0,$$

with non-negative integers $m_1, \ldots, m_n$ that are not all zero.

Remark 4.6. Note that if $n \geq 4$, then the $H_{k_i k_j}$'s are the only codimension 2 subvarieties Z of X that are locally complete intersections whose ideal sheaves I_Z admit projective resolutions of the form $0 \to L \to V \to I_Z \to 0$, where L is a line bundle and V is a rank-2 vector bundle on X. When $n = 3$, one also has $Z = H_{k_i k_j} + H_{k_i k_l}$, with $k \neq l$.

Proof. The proof of (i) and (ii) is the same as that of Proposition 3.3. We therefore assume that E is not an extension of line bundles. Since all vector bundles on X are filtrable, it can then be written in the form $E = L_a \otimes E'$, where E' is a locally free extension of the form

$$0 \to \mathcal{O} \to E' \to L_\beta \otimes I_Z \to 0, \tag{4.7}$$

I_Z is the ideal sheaf of a codimension 2 subvariety Z of X that is a locally complete intersection, and $a, \beta \in \mathbb{C}^*$. By Remark 4.6, we must have $Z = H_{k_i k_j}$ or $H_{k_i k_{j_1}} + H_{k_i k_{j_2}}$, for some $k_i, k_{j_1}, k_{j_2} \geq 1$. We therefore have to determine which extensions of the form (4.7) give rise to non-isomorphic rank-2 vector bundles: this is done by analysing the exact sequence

(4.8)
$$0 \to H^1(X, L_{\beta^{-1}}) \to \mathrm{Ext}^1(X; I_Z, L_{\beta^{-1}}) \to H^0(X, Ext^1(I_Z, L_{\beta^{-1}})) \to H^2(X, L_{\beta^{-1}}).$$

Recall that an extension class $\xi \in \mathrm{Ext}^1(X; I_Z, L_{\beta^{-1}})$ determines a locally free sheaf if and only if its image in $H^0(X, Ext^1(I_Z, L_{\beta^{-1}}))$ generates the stalk of $Ext^1(I_Z, L_{\beta^{-1}})$ at every point of X. Since $Ext^1(I_Z, L_{\beta^{-1}})$ is a torsion sheaf supported on each $H := H_{k_i k_j}$ appearing in Z, a necessary condition for the existence of a locally free extension is therefore that $h^0(H, Ext^1(I_Z, L_{\beta^{-1}})|_H) \neq 0$ for each H. Note that the restriction of $Ext^1(I_Z, L_{\beta^{-1}})$ to H can be identified with the line bundle $\det(\mathcal{N}_{H/X}) \otimes L_{\beta^{-1}}$, which is isomorphic to $L_{\beta^{-1}\mu_i^{k_i}\mu_j^{k_j}}$ (because $\mathcal{N}^*_{H/X} \cong I_H/I_H^2$ is generated as an $\mathcal{O}_H$-module by $z_i^{k_i}$ and $z_j^{k_j}$).

Let us first assume that $n = 3$. Note that in this case the underlying space of H is the elliptic curve $\mathbb{C}^*/\mu_k$ with $k \neq i, j$; consequently, a global section of $L_{\beta^{-1}\mu_i^{k_i}\mu_j^{k_j}}$ on H is given by a holomorphic function $g(z)$ such that

$$g(\mu z) = \beta^{-1}\mu_i^{k_i}\mu_j^{k_j} g(z)$$

whose Laurent series expansion is of the form $\sum_{t=-\infty}^{\infty} \sum_{s=0}^{k_j-1} \sum_{r=0}^{k_i-1} a_{rst} z_i^r z_j^s z_k^t$. One can easily verify that such a function exists if and only if

$$\beta = \mu_i^{m_i} \mu_j^{m_j} \mu_k^{-\nu},$$

where m_i, m_j, ν are integers such that $1 \leq m_l \leq k_l$ for $l = i, j$, so that

$$g(z) = a_0 z_i^{k_i - m_i} z_j^{k_j - m_j} z_k^{-\nu}$$

for some $a_0 \in \mathbb{C}$. Thus, $h^0(H, Ext^1(I_Z, L_{\beta^{-1}})|_H) \neq 0$ if and only if β satisfies (4.9), in which case $h^0(H, Ext^1(I_Z, L_{\beta^{-1}})|_H) = 1$. However, the function $g(z)$ must also generate the stalk of $Ext^1(I_Z, L_{\beta^{-1}})$ at every point, implying that we must have $m_i = k_i$ and $m_j = k_j$. A necessary condition for the existence of a locally free extension is therefore that

$$\beta = \mu_i^{k_i} \mu_j^{k_j} \mu_k^{-\nu}, \tag{4.9}$$

for some integer ν.

Let us first consider the case $Z = H_{k_i k_j}$. Referring to (4.9), this means that $\beta = \mu_i^{k_i} \mu_j^{k_j} \mu_k^{-\nu}$. We have to determine which values of ν give rise non-isomorphic rank-2 vector bundles. We shall see that there are two different cases depending on whether or not ν is positive. If $\nu > 0$, then $h^q(X, L_{\beta^{-1}}) = 0$ for $q = 1, 2$, so that (4.8) reduces to:

$$\mathrm{Ext}^1(I_Z, L_{\beta^{-1}}) = H^0(X, Ext^1(I_Z, L_{\beta^{-1}})) = \mathbb{C}.$$

Therefore, the extension determined by a non-zero element of $\mathrm{Ext}^1(I_Z, L_{\beta^{-1}})$ determine, up to isomorphism, a unique rank-2 vector bundle. Note that this bundle cannot be an extension of line bundles.

If $\nu \leq 0$, although $h^1(X, L_{\beta^{-1}}) = 0$, we have $h^2(X, L_{\beta^{-1}}) \neq 0$. Let us assume that a class ξ in $\mathrm{Ext}^1(I_Z, L_{\beta^{-1}})$ generates a rank-2 vector bundle E'. In this case, $\mathcal{O}$ is not a destabilising line bundle of E'. Indeed, given the exact sequence

$$0 \to L_{\mu_i^{-k_i} \mu_j^{-k_j}} \to L_{\mu_i^{-k_i}} \oplus L_{\mu_j^{-k_j}} \to I_H \to 0, \tag{4.10}$$

one verifies that $L_{\mu_i^{k_i} \mu_k^{-\nu}}$ and $L_{\mu_j^{k_j} \mu_k^{-\nu}}$ not only map non-trivially to $L_{\mu_i^{k_i} \mu_j^{k_j} \mu_k^{-\nu}} \otimes I_H$, but also to E'; furthermore, they are the line bundles of maximal degree mapping into E'. Consequently, the quotient sheaves $E'/L_{\mu_i^{k_i} \mu_k^{-\nu}}$ and $E'/L_{\mu_j^{k_j} \mu_k^{-\nu}}$ are torsion free, so that, for example, we have an exact sequence

$$0 \to \mathcal{O} \to L_{\mu_i^{-k_i} \mu_k^{\nu}} \otimes E' \to L_{\mu_i^{-k_i} \mu_j^{k_j} \mu_k^{\nu}} \otimes I_{Z'} \to 0,$$

where Z' is either empty or a locally complete intersection of codimension 2 in X. If $\nu = 0$, then $E' = L_{\mu_i^{k_i}} \oplus L_{\mu_j^{k_j}}$, implying that Z' is empty. However, if $\nu < 0$, then $\beta' = \mu_i^{-k_i} \mu_j^{k_j} \mu_k^{\nu}$ does not satisfy condition (4.9), leading to a contradiction. Hence, we get locally free sheaves only for $\nu > 0$, in which case they are not extensions of line bundles.

Let us now assume that $Z = H_{k_i k_{j_1}} + H_{k_i k_{j_1}}$. In this case, since β must satisfy (4.9) for both (k_i, k_{j_1}) and (k_i, k_{j_2}), we have $\beta = \mu_i^{k_i} \mu_j^{k_{j_1}} \mu_k^{k_{j_2}}$. As above, one can show that E' decomposes as $L_{\mu_i^{k_i}} \oplus L_{\mu_j^{k_{j_1}} \mu_k^{k_{j_2}}}$. The only extensions giving rank-2 vector bundles that are not extensions of line bundles therefore correspond to codimension 2 subvarieties of the form $Z = H_{k_i k_j}$. This proves (iii) for $n = 3$.

Finally, let us assume that $n \geq 4$ so that the underlying space of H is a generic Hopf manifold of dimension at least 2. As in the case $n = 3$, we can easily show that there exists a locally free extension E' if and only if

$$\beta = \mu_1^{-m_1} \dots \mu_i^{k_i} \dots \mu_j^{k_j} \dots \mu_n^{-m_n},$$

for non-negative integers $m_1, \dots, m_n$; note that the m_l's must all be non-negative because H is now a Hopf manifold. Moreover, E' is not an extension of line bundles, unless all the m_l's are zero, in which case $E' = L_{\mu_i^{k_i}} \oplus L_{\mu_j^{k_j}}$. □

We finish the paper by determining stability conditions for holomorphic rank-2 vector bundles on higher dimensional Hopf manifolds.

Theorem 4.11. *Let E be a rank-2 vector bundle on a generic Hopf manifold X of dimension greater than 2.*

(i) If E is an extension of line bundles, then it is unstable.

(ii) Otherwise, $E = L_a \otimes E'$ for $a \in \mathbb{C}^*$ and a non-trivial extension E' of $I_{H_{k_i k_j}}$ by $L_{\mu_1^{m_1} \dots \mu_i^{-k_i} \dots \mu_j^{-k_j} \dots \mu_n^{m_n}}$, where the non-negative integers $m_1, \dots, m_n$ are not all zero (see Proposition 4.5 (iii)). In this case, E is stable if and only if

$$\prod_{\substack{0 \leq l \leq n \\ l \neq i,j}} |\mu_l^{m_l}| > |\mu_i^{k_i} \mu_j^{k_j}|.$$

Note that one can always find integers $m_1, \dots, m_n$ that satisfy this equation. Consequently, stable rank-2 vector bundles exist on X. □

Proof. We begin by determining the destabilising line bundles of rank-2 vector bundles; given that decomposable bundles are automatically unstable, we only consider the indecomposable case. Let E be an indecomposable rank-2 vector bundle on X. Suppose that $E = L_a \otimes E'$, where $a \in \mathbb{C}^*$ and E' is one of the non-trivial extension $0 \to L \to E' \to I_Z \to 0$ of Proposition 4.5 (ii) or (iii). Therefore, any line bundle mapping non-trivially into E is of the form

$$L \otimes L_{a\mu_1^{-l_1} \dots \mu_n^{-l_n}},$$

for non-negative integers $l_1, \dots, l_n$. This is obvious when Z is empty. Otherwise, if $Z = H_{k_i k_j}$, then this comes from the fact that every bundle mapping to $I_{H_{k_i k_j}}$ also maps to L. Consequently, since $L \otimes L_a$ is the unique destabilising line bundle of E, the theorem follows from the definition of degree. □

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